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Cyclotomic Polynomial

A polynomial given by

$$\Phi_n(x) = \prod_{k=1}^{n'} (x - \zeta_k),$$

where ζ_k are the roots of unity in $\mathbb{C}$ given by

$$\zeta_k \equiv e^{2\pi i k/n}$$

and k runs over integers relatively prime to n. The prime may be dropped if the instead taken over primitive roots of unity, so that

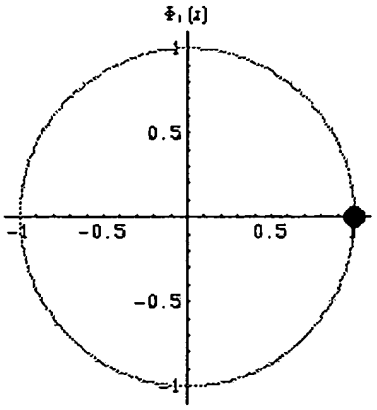
$$\Phi_n(x) = \prod_{\substack{k=1 \\ \text{primitive } \zeta_k}}^{n} (x - \zeta_k).$$

The notation $F_n(x)$ is also frequently encountered. Dickson *et al.* (1923) and I (1975) give extensive bibliographies for cyclotomic polynomials.

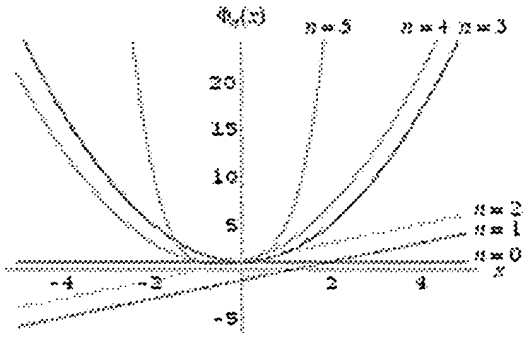
The cyclotomic polynomial for $n > 1$ can also be defined as

$$\Phi_n(x) = \prod_{d|n} (1 - x^{n/d})^{\mu(d)}$$

where $\mu(d)$ is the Möbius function and the product is taken over the divisors d 1991, p. 225).



$\Phi_n(x)$ is an integer polynomial and an irreducible polynomial with polynomial $\phi(n)$, where $\phi(n)$ is the totient function. Cyclotomic polynomials are returned *Mathematica* command `Cyclotomic[n, x]`. The roots of cyclotomic polynomials l unit circle in the complex plane, as illustrated above for the first few cyclotomic polynomials.



The first few cyclotomic polynomials are

$$\Phi_1(x) = x - 1$$

$$\Phi_2(x) = x + 1$$

$$\Phi_3(x) = x^2 + x + 1$$

$$\Phi_4(x) = x^2 + 1$$

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$$

$$\Phi_6(x) = x^2 - x + 1$$

$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

$$\Phi_8(x) = x^4 + 1$$

$$\Phi_9(x) = x^6 + x^3 + 1$$

$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1.$$

If p is an odd prime, then

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + x + 1$$

$$\Phi_{2p}(x) = \frac{x^{2p} - 1}{x^p - 1}\frac{x - 1}{x^2 - 1} = x^{p-1} - x^{p-2} + \ldots - x + 1$$

$$\Phi_{4p}(x) = \frac{x^{4p} - 1}{x^{2p} - 1}\frac{x^2 - 1}{x^4 - 1} = x^{2p-2} - x^{2p-4} + \ldots - x^2 + 1$$

(Riesel 1994, p. 306). Similarly, for p again an odd prime,

$$x^p - 1 = \Phi_1(x)\Phi_p(x)$$

$$x^{2p} - 1 = \Phi_1(x)\Phi_2(x)\Phi_p(x)\Phi_{2p}(x)$$

$$x^{4p} - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_p(x)\Phi_{2p}(x)\Phi_{4p}(x).$$

For the first few remaining values of n,

$$x - 1 = \Phi_1(x)$$

$$x^2 - 1 = \Phi_1(x)\Phi_2(x)$$

$$x^4 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)$$

$$x^8 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x)$$

$$x^9 - 1 = \Phi_1(x)\Phi_3(x)\Phi_9(x)$$

$$x^{15} - 1 = \Phi_1(x)\Phi_3(x)\Phi_5(x)\Phi_{15}(x)$$

$$x^{16} - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x)\Phi_{16}(x)$$

$$x^{18} - 1 = \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_6(x)\Phi_9(x)\Phi_{18}(x)$$

(Riesel 1994, p. 307).

For p a prime relatively prime to n,

$$F_{np}(x) = \frac{F_n(x^p)}{F_n(x)},$$

but if $p|n$,

$$F_{np}(x) = F_n(x^p)$$

(Nagell 1951, p. 160).

An explicit equation for $\Phi_n(x)$ for squarefree *n* is given by

$$\Phi_n(x) = \sum_{j=0}^{\phi(n)} a_{nj} x^{\phi(n)-j},$$

where a_{nj} is calculated using the recurrence relation

$$a_{nj} = -\frac{\mu(n)}{j} \sum_{m=0}^{j-1} a_{nm} \mu(\mathrm{GCD}(n, j-m)) \phi(\mathrm{GCD}(n, j-m)),$$

with $a_{n0} = 1$, where $\mu(n)$ is the Möbius function and $\mathrm{GCD}(m, n)$ is the gre common denominator of *m* and *n*.

The polynomial $x^n - 1$ can be factored as

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where $\Phi_d(x)$ is a cyclotomic polynomial. Furthermore,

$$x^n + 1 = \frac{x^{2n} - 1}{x^n - 1} = \frac{\prod_{d|2n} \Phi_d(x)}{\prod_{d|n} \Phi_d(x)}.$$

The coefficients of the inverse of the cyclotomic polynomial

$$\frac{1}{1 + x + x^2} = 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \ldots$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

can also be computed from

$$\begin{aligned} c_n &= 1 - 2\left\lfloor \tfrac{1}{3}(n+2) \right\rfloor + \left\lfloor \tfrac{1}{3}(n+1) \right\rfloor + \left\lfloor \tfrac{1}{3} n \right\rfloor \\ &= 1 - 3\left\lfloor \tfrac{1}{3}(n+2) \right\rfloor + \lfloor n \rfloor \\ &= \frac{2}{\sqrt{3}} \sin\left[\tfrac{2}{3}\pi(n+1)\right], \end{aligned}$$

where $\lfloor x \rfloor$ is the floor function.

For p prime,

$$\Phi_p(x) = \sum_{k=0}^{p-1} x^k,$$

i.e., the coefficients are all 1. The first cyclotomic polynomial to have a coefficie than ± 1 and 0 is $\Phi_{105}(x)$, which has coefficients of -2 for x^7 and x^{41}. This i because 105 is the first number to have three distinct odd prime factors, i.e., $105 = 3 \cdot 5 \cdot 7$ (McClellan and Rader 1979, Schroeder 1997). The smallest val which $\Phi_n(x)$ has one or more coefficients ± 1, ± 2, ± 3, ... are 0, 105, 385, 1 2805, 3135, 6545, 6545, 10465, 10465, 10465, 10465, 10465, 11305, ... (Slo A013594).

It appears to be true that, for $m, n > 1$, if $\Phi_m(x) + \Phi_n(x)$ factors, then the contain a cyclotomic polynomial. For example,

$$\Phi_7(x) + \Phi_{22}(x) = (x^2 + 1)(x^8 - x^7 + 2x^4 + 2) = \Phi_4(x)(x^8 - x^7 + 2x^4$$

This observation has been checked up to $m, n = 150$ (C. Nicol). If m and n ar then $C_m + C_n$ is irreducible.

Migotti (1883) showed that coefficients of $\Phi_{pq}(x)$ for p and q distinct primes c 0, ± 1. Lam and Leung (1996) considered

$$\Phi_{pq}(x) = \sum_{k=0}^{pq-1} a_k x^k$$

for p, q prime. Write the totient function as

$$\phi(pq) = (p-1)(q-1) = rp + sq$$

and let

$$0 \le k \le (p-1)(q-1),$$

then

1. $a_k = 1$ iff $k = ip + jq$ for some $i \in [0, r]$ and $j \in [0, s]$,
2. $a_k = -1$ iff $k + pq = ip + jq$ for $i \in [r+1, q-1]$ and $j \in [s+$
3. otherwise $a_k = 0$.

The number of terms having $a_k = 1$ is $(r+1)(s+1)$, and the number of te $a_k = -1$ is $(p-s-1)(q-r-1)$. Furthermore, assume $q > p$, then the m

coefficient of Φ_{pq} is $(-1)^r$.

Resultants of cyclotomic polynomials have been computed by Lehmer (1930), D (1940), and Apostol (1970). It is known that $\rho(\Phi_k(x), \Phi_n(x)) = 1$ if (m, n) m and n are relatively prime (Apostol 1975). Apostol (1975) showed that for po integers m and n and arbitrary nonzero complex numbers a and b,

$$\rho(\Phi_m(ax), \Phi_n(bx)) = b^{\phi(m)\phi(n)} \prod_{d|n} \left[\Phi_{m/\delta}\left(\frac{a^d}{b^d}\right)\right]^{\mu(n/d)\phi(m)/\phi(m/\delta)}$$

where $\delta = \mathrm{GCD}(m, d)$ is the greatest common divisor of m and d, $\phi(n)$ is tl function, $\mu(n)$ is the Möbius function, and the product is over the divisors of n are distinct primes p and q, then (44) simplifies to

$$\rho(\Phi_q(ax), \Phi_p(bx)) = \begin{cases} \frac{a^{pq}-b^{pq}}{a^p-b^p}\frac{a-b}{a^q-b^q} & \text{for } a \neq b \\ a^{(p-1)(q-1)} & \text{for } a = b. \end{cases}$$

The following table gives the resultants $\rho(\Phi_k(x), \Phi_n(x))$ (Sloane's A054372).

$k\backslash n$	1	2	3	4	5	6	7
1	0						
2	2	0					
3	3	1	0				
4	2	2	1	0			
5	5	1	1	1	0		
6	1	3	4	1	1	0	
7	7	1	1	1	1	1	0

The numbers of 1s in successive rows of this table are given by 0, 0, 1, 1, 3, 3, 9, ... (Sloane's A075795).

The cyclotomic polynomial $\Phi_6(x)$ has the particularly nice Maclaurin series

$$\frac{1}{\Phi_6(x)} = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \ldots,$$

whose coefficients 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, ... (Sloane's A010892) are giv solving the recurrence equation

$$a(n) = a(n-1) - a(n-2)$$

with $a(0) = a(1) = 1$ (Wolfram 2002, p. 128), giving the explicit form

$$a(n) = \frac{2}{3}\sqrt{3}\sin[\tfrac{1}{3}(n+1)\pi].$$

Interestingly, any sequence $b(n)$ satisfying the linear recurrence equation

$$b(n) = b(n-1) - b(n-2)$$

can be written as

$$b(n) = b(0)a(n) + [b(1) - b(0)]a(n-1).$$

SEE ALSO: Aurifeuillean Factorization, Gauss's Cyclotomic Formula, Lucas's Theor Inversion Formula, Primitive Root of Unity, Root of Unity

RELATED WOLFRAM SITES: http://functions.wolfram.com/Polynomials/Cyclot

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